Influence of shear flow on the Fréedericksz transition in nematic liquid crystals

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Within the framework of Ericksen-Leslie continuum theory we analyze the influence of shear flow on the magnetic-field-induced Fréedericksz transition in nematic liquid crystal with rodlike molecules. We consider three basic orientational configurations of a nematic planar layer in the uniform magnetic field. Conditions of rigid director coupling on the boundaries of the layer and constant shear flow gradient inside the layer are used. We exhibit some flow aligning effects for nematic liquid crystals with various ratio of rotary viscosities and investigate how unequal elastic constants (elastic anisotropy) alter the magnetic Fréedericksz transition in sheared nematics. Our calculations predict that surface boundary effects in nematic films and magnetic field action lead to existence of stationary flow regimes in the so-called nonflow aligning nematics, otherwise, surface and magnetic forces extend the range of viscous coefficient values corresponding to the flow aligning regimes. We show that imposing of shear flow on the Fréedericksz transition leads to a threshold behavior or to a "smoothing" of the transition. It depends on the orientation of the nematic layer in magnetic field and magnitudes of rotary viscous coefficients.

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I. INTRODUCTION

One of the most interesting effects in the physics of liquid crystals from both the fundamental and practical points of view is the Fréedericksz transition [1,2]. In nematic liquid crystal (NLC) the intermolecular interactions responsible for the nematic order tend to orient long axes of molecules parallel to a common direction **n**, called the director. The director **n** is uniform throughout the sample in the absence of external fields or certain boundary conditions. A sufficiently strong magnetic field applied across a planar nematic layer for the anisotropy of the diamagnetic susceptibility $\chi_a > 0$ tends to orient the director in the direction of field. If the field is perpendicular to the unperturbed director alignment the distortions of orientation take place above a critical field, i.e., the Fréedericksz transition occurs. The equilibrium orientation of the director in the layer is determined by the interaction of liquid crystal (LC) with the applied magnetic field, the elastic LC forces and surface boundary effects. The Fréedericksz transition underlies of numerous LC display devices, moreover, deformations of LC structure caused by external fields are employed in experimental techniques to determine LC elastic constants [3,4].

The static Fréedericksz transition as a bifurcation problem was first discussed by Derfel [5]. Using methods of catastrophe theory, he reanalyzed a series of static effects discovered earlier by other authors. The author [5] considered a nematic layer with pretilted director orientation in obliquely applied magnetic field. He performed qualitative analysis of hysteresis effects caused by a symmetry breaking in a system. Blake *et al.* [6] considered the effects of imperfections emergent in the Fréedericksz transition. These imperfections are produced by special orientation of a magnetic field relative to the nematic layer or misalignment of the director at the boundaries. Derfel in Ref. [7] revealed the analogy between the magnetic-field-induced deformations and flow-induced deformations in liquid crystals. He predicted new threshold effects in sheared, weakly anchored nematics.

The purpose of the present paper is to analyze the combined effect of shear flow and magnetic field on the nematic orientational structure. We study the influence of shear flow on the magnetic-field-induced Fréedericksz transition in a planar layer of a nematic liquid crystal with rodlike molecules. Linear approximation of a velocity profile and rigid coupling of the director on the plates bounding the NLC layer are used. We consider three basic orientations of the layer in the uniform magnetic field. In all studied configurations the external magnetic field counteracted to the surface forces aligning the director on the boundaries. We reveal some flow aligning effects for the nonflow aligning nematics and investigate how unequal elastic constants change the director deformations.

Recently, the Fréedericksz transition as a bifurcation at the presence of shear flow was considered by Mukherjee *et al.* [8]. They made assumptions strongly restricted to the area of applicability of the solutions and some of them were not correct. The authors [8] used the so-called one-constant approximation for the Frank modules (elastic isotropy), but real nematics have unequal elastic modules (elastic anisotropy), and so the assumption of elastic isotropy is unrealistic. In the present paper, we take into account the anisotropy of Frank modules. The authors [8] have also supposed that the rotary viscous coefficient γ_1 and the irrotational torque coefficient γ_2 have the same value and sign. Due to positiveness of entropy production the viscous coefficient $\gamma_1 > 0$, but γ_2 can have any sign [1,2], nevertheless, for all known rodlike LCs the coefficient $\gamma_2 < 0$ [1–4,9]. The case $\gamma_2 > 0$ considered in Ref. [8] concerns, apparently, the disklike nematics [10–14]. Furthermore, the authors [8] have considered the planar approximation for the director field in the case of its three-dimensional deformations caused by magnetic field and shear flow [see configuration (B) in Ref. [8]]. Their solution

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does not satisfy the equations of the NLC motion for this configuration.

In the present paper, we employ the values of viscous coefficients, which are suitable for the planar director field of NLC with rodlike molecules in all basic orientations of the NLC layer in magnetic field. We analyze the role of nematic elastic anisotropy and viscous anisotropy on the nematic ordering in magnetic-field-induced Fréedericksz transition under shear flow. We predict that surface boundary effects in LC layer and magnetic field action lead to existence of stationary flow regimes in the so-called nonflow aligning nematics. We show that imposing of shear flow on the Fréedericksz transition leads to a threshold behavior or to a "smoothing" of the transition.

The paper is organized as follows. In Sec. II, we summarize the nematodynamic equations of the Ericksen-Leslie continuum theory. In Sec. III, we discuss the effect of shear flow on the Fréedericksz transition for the basic orientations of the nematic layer in magnetic field. Our conclusions are presented in Sec. IV.

II. NEMATODYNAMIC EQUATIONS

In order to describe the dynamics of a liquid crystal we use the continuum theory proposed by Ericksen and Leslie [1,2,15]. We consider shear flow of an incompressible nematic liquid crystal with the velocity $\mathbf{v}$ and the director $\mathbf{n}$ fields varying in time and space.

The density of the bulk free energy of a nematic liquid crystal in magnetic field for the small isothermal deformations is given by [1]

$$F = F_d + F_m,$$

$$F_d = \tfrac{1}{2}[K_1(\nabla\mathbf{n})^2 + K_2(\mathbf{n}\cdot\nabla\times\mathbf{n})^2 + K_3(\mathbf{n}\times\nabla\times\mathbf{n})^2],$$

$$F_m = -\tfrac{1}{2}\chi_a(\mathbf{n}\cdot\mathbf{H})^2. \tag{1}$$

Here K_1, K_2, K_3 are the splay, twist, and bend modules of NLC orientational elasticity, known as Frank elastic constants, $\mathbf{n}$ is the director of a liquid crystal, χ_a is the anisotropy of a magnetic susceptibility, $\mathbf{H}$ is the external magnetic field strength.

The first term F_d in Eq. (1) represents the bulk free energy of the director field elastic deformations (the Oseen-Frank potential), and the following one F_m in (1) is the bulk free energy of a magnetic field interaction with a nematic.

According to the Ericksen-Leslie continuum theory [1,2,15] the equations of NLC motion can be written as

$$\rho\frac{d\mathbf{v}}{dt} = \nabla\cdot\bar{\bar{\boldsymbol{\sigma}}}, \tag{2}$$

$$\nabla\cdot\mathbf{v} = 0, \tag{3}$$

$$\mathbf{h} = \gamma_1\mathbf{N} + \gamma_2\mathbf{n}\cdot\bar{\bar{\mathbf{A}}}. \tag{4}$$

Here ρ and $\mathbf{v}$ are the mass density and the velocity of fluid, respectively; d/dt denotes the convective time derivative $\partial/\partial t + \mathbf{v}\nabla$. Equations (2)–(4) present the balance of forces acting on the fluid, the incompressibility condition and the balance of torques acting on the director $\mathbf{n}$, respectively.

The stress tensor $\bar{\bar{\boldsymbol{\sigma}}}$ in Eq. (2) is determined as a sum

$$\bar{\bar{\boldsymbol{\sigma}}} = \bar{\bar{\boldsymbol{\sigma}}}' + \bar{\bar{\boldsymbol{\sigma}}}^e, \tag{5}$$

where viscous stress tensor $\bar{\bar{\boldsymbol{\sigma}}}'$ has the form

$$\sigma'_{ki} = \alpha_1 n_k n_i n_l n_m A_{lm} + \alpha_2 n_k N_i + \alpha_3 n_i N_k + \alpha_4 A_{ki} + \alpha_5 n_k n_l A_{li} + \alpha_6 n_i n_l A_{lk},$$

here the summation over repeated indices is implied. The vector $\mathbf{N} = d\mathbf{n}/dt - \bar{\bar{\boldsymbol{\omega}}}\cdot\mathbf{n}$ represents the rate of change of the director relative to the background liquid, $A_{ik} = (1/2)(\partial_k v_i + \partial_i v_k)$ and $\omega_{ik} = (1/2)(\partial_k v_i - \partial_i v_k)$ are symmetric and antisymmetric parts of the velocity gradients tensor, respectively. The six viscosity coefficients α_s are called Leslie coefficients. Only five of them are independent due to the relationship first derived by Parodi [1,2], $\alpha_2 + \alpha_3 = \alpha_6 - \alpha_5$. The correlation of rotary viscosity coefficients γ_1 and γ_2 with Leslie coefficients defined by Onsager reciprocal relation may be written as $\gamma_1 = \alpha_3 - \alpha_2$, $\gamma_2 = \alpha_3 + \alpha_2$. The coefficient γ_1 characterizes the viscous torque associated with an angular velocity of the director, while γ_2 gives the contribution to this torque due to a shear velocity in LC.

The elastic part of stress tensor (5) known as Ericksen tensor $\bar{\bar{\boldsymbol{\sigma}}}^e$ is given by

$$\sigma_{ki}^{(e)} = -p\delta_{ki} - \frac{\partial F}{\partial(\partial_k n_l)}\partial_i n_l,$$

where p is the pressure, δ_{ki} is the Kronecker symbol, and $\partial_i \equiv \partial/\partial x_i$.

The vector of the molecular field $\mathbf{h}$ in the equation of the director motion (4) includes elastic and magnetic-field terms and can be expressed in the following form:

$$h_i = -\frac{\partial F}{\partial n_i} + \partial_k\frac{\partial F}{\partial(\partial_k n_i)}.$$

Thus, Eqs. (2)–(4) represent the total set of the nematodynamic equations.

III. SHEAR FLOW INFLUENCE ON THE FRÉEDERICKSZ TRANSITION

We consider the NLC layer of thickness d, sandwiched between two parallel plates moving relatively to each other. We introduce rectangular coordinate system with the x axis directed along the trajectory of the moving plate and the z axis perpendicular to the plates. The coordinate origin is chosen in the middle of the layer. The layer bounded at $z=-d/2$ and $z=d/2$ has an infinite extension in the x and y directions (see Figs. 1, 7, and 8). The lower plate is at rest in our coordinate frame while the upper one moves with constant velocity $\mathbf{U}$. We consider the stationary solutions of Eqs. (2)–(4) corresponding to the three basic types of NLC layer arrangement in the uniform magnetic field $\mathbf{H}$ (Figs. 1, 7, and

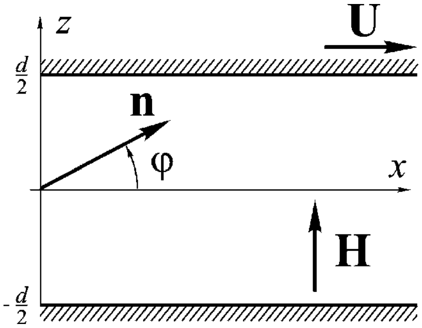

FIG. 1. Orientation of a nematic layer between moving plates in magnetic field $\mathbf{H}$ [configuration (A)].

8); we suppose also that anisotropy of a diamagnetic susceptibility χ_a is positive, hence the director $\mathbf{n}$ tends to align in the field direction.

A. Configuration (A)

Let the magnetic field $\mathbf{H}=(0,0,H)$ be directed perpendicularly to the plates of the layer and the velocity of nematic flow has the form $\mathbf{v}=(v_x(z),0,0)$ (see Fig. 1). We assume rigid planar coupling of the director on the boundaries of the layer. In the absence of the magnetic field and the flow, the director is uniform and directed along the x axis in the layer. If we impose the shear flow and magnetic field to the layer the perturbed director $\mathbf{n}$ in this configuration can be written as

$$\mathbf{n}=(\cos\varphi(z),0,\sin\varphi(z)), \tag{6}$$

where $\varphi(z)$ is the tilt angle of the director $\mathbf{n}$ from the x axis.

We use the linear approximation of the velocity field $\mathbf{v}=(Uz/d,0,0)$ in the layer [8,16], here U is the velocity of the upper plate. In the real situation a transverse flow $(v_y\neq 0)$ exists and the gradient of shear rate $\partial v_x/\partial z$ depends on z. We disregard these effects which would make the calculations much more complicated.

We choose the thickness d of the layer as the unit of length and introduce the dimensionless coordinate $\tilde{z}=z/d$. Stationary $(\partial\mathbf{n}/\partial t=0)$ equation of the director motion (4) with the help of Eq. (6) can be expressed as

$$f_A(\varphi)\frac{d^2\varphi}{d\tilde{z}^2}+\frac{1}{2}\frac{df_A(\varphi)}{d\varphi}\left(\frac{d\varphi}{d\tilde{z}}\right)^2=-h\sin 2\varphi+\mu(1-\gamma\cos 2\varphi) \tag{7}$$

with boundary conditions

$$\varphi(-1/2)=\varphi(1/2)=0, \tag{8}$$

where $f_A(\varphi)=\cos^2\varphi+k\sin^2\varphi$. Here we introduce the following dimensionless parameters: $h=\chi_a H^2 d^2/(2K_1)$, $\mu=U\gamma_1 d/(2K_1)$, $k=K_3/K_1$, and $\gamma=-\gamma_2/\gamma_1$. The parameter h represents the square of dimensionless magnetic field strength, μ is the dimensionless gradient of NLC shear rate known as Ericksen number, which is the ratio of the viscous $(\sim U\gamma_1/d)$ to elastic forces $(\sim K_1/d^2)$. The coefficient k characterizes the anisotropy of Frank elastic constants, γ is the reactive parameter. The case $\gamma\geqslant 1$ corresponds to the so-called flow aligning liquid crystals. In these materials the director tends to align inside the shear plane at the angle defined by relation [1,2]

$$\cos 2\varphi_L=1/\gamma, \tag{9}$$

where φ_L is the flow alignment angle, known as Leslie angle, which determines possible planar solutions for NLC director field in the absence the external forces and boundary effects (the unbounded NLC). For the case $0\leqslant\gamma<1$ there is no solutions of Eq. (9) and this inequality defines nonflow aligning liquid crystals, which have no steady state orientation of the director in the shear plane. Relation (9) follows from Eq. (7) for the uniform $(\partial\varphi/d\tilde{z}=0)$ director configuration in the magnetic field absence.

Under the influence of a magnetic field the uniform solution of Eq. (7) is given by

$$\tan\varphi=\frac{h+\sqrt{h^2+\mu^2(\gamma^2-1)}}{\mu(1+\gamma)}, \tag{10}$$

which has been received earlier in Ref. [15]. In the magnetic field absence $(h=0)$ Eq. (10) reduces to Eq. (9). As it is seen from Eq. (10) the stationary planar solutions exist for γ values determined by the inequality $\gamma^2\geqslant 1-(h/\mu)^2$. It means that in the unbounded NLC subjected to the magnetic field and shear flow the uniform solutions exist for wider range of γ values than in sheared nematics solely.

It should be pointed out that in the absence of flow $(\mu=0)$ and magnetic field $(h=0)$ Eq. (7) has the uniform solution $\varphi=0$ satisfying boundary conditions (8). As it is seen from Eq. (7), this solution is valid for arbitrary h without shear flow or with shear flow for $\gamma=1$.

We consider the nonuniform solutions for a director field. For that purpose we multiply Eq. (7) by $d\varphi/d\tilde{z}$ and integrate it twice. Then the stationary solutions describing the perturbed state of the director in the middle of the layer can be found from the following integral equation:

$$\frac{1}{2}=\pm\int_0^{\varphi_0}\sqrt{\frac{f_A(\varphi)}{\Phi_A(\varphi,\varphi_0)}}d\varphi, \tag{11}$$

where $\Phi_A(\varphi,\varphi_0)=h(\cos 2\varphi-\cos 2\varphi_0)+2\mu(\varphi-\varphi_0)+\mu\gamma\times(\sin 2\varphi_0-\sin 2\varphi)$. Here $\varphi_0=\varphi(0)$ is the angle of the director rotation in the middle of the layer. The plus sign in Eq. (11) corresponds to positive values of φ_0 and the minus sign to negative ones. In the absence of flow Eq. (11) is invariant under the replacement $\varphi\to-\varphi$, i.e., positive and negative solutions are symmetrical under this condition.

Let us estimate the magnitudes of h and μ for typical values of NLC parameters. Assuming $\chi_a\approx 10^{-7}$ SGSE units, $K_1\approx 10^{-6}$ dynes, the rotary viscous coefficients $\gamma_1\approx 10^{-1}$ poise and choosing the thickness of the layer $d\approx 10^{-3}$ sm, the velocity of the upper plate $U\approx 10^{-2}$ sm/s and the magnetic field strength $H\approx 10^4$ Oe, we obtain $h\approx 10$ and $\mu\approx 1$.

1. Critical magnetic field of a transition

For $\gamma=1$ Eq. (7) has the trivial solution $\varphi=0$ corresponding to uniform nematic state with the director orientation along the x axis. Such a solution, however, becomes unstable in the field h exceeding some threshold value h_c. Close to a bifurcation point $\varphi \ll 1$ and so it is possible to look for a solution of Eq. (7) as the expansion in terms of the small parameter describing the proximity of a magnetic field strength h to the critical value h_c. In the low order approximation with the help of boundary conditions (8) we obtain

$$h_c = \pi^2/2, \tag{12}$$

$$\varphi(\tilde{z}) = \frac{3\pi}{8\mu}(h - h_c)\cos \pi\tilde{z}. \tag{13}$$

Expression (13) characterizes the behavior of the angle of the director orientation near the phase transition point from the uniform state into the distorted one. The symmetry of Eq. (7) solutions is broken under the shear flow influence. As it is seen from Eq. (13), the derivative $\partial\varphi/\partial h$ is positive and finite at the transition point (see Figs. 3 and 4). Besides, the quantity h_c coincides with a square of the critical value of dimensionless magnetic field strength in static Fréedericksz transition, which in dimension form is given by [1,2]

$$H_c = \frac{\pi}{d}\sqrt{\frac{K_1}{\chi_a}}.$$

2. Weak shear flow and magnetic field

The trivial solution $\varphi = 0$ of Eq. (7) is absent at $\mu(1-\gamma) \neq 0$. For small values of this parameter, for example, at $\mu \ll 1$ (weak flow) or $\gamma \approx 1$, when φ is small, Eq. (7) is possible to linearize

$$\varphi'' + 2h\varphi = \mu(1-\gamma).$$

Hereinafter the prime denotes $\tilde{z}$-coordinate derivative. The general solution of this equation satisfying boundary conditions (8) becomes

$$\varphi(\tilde{z}) = \frac{\mu(\gamma-1)}{2h}\left(\frac{\cos(\sqrt{2h}\tilde{z})}{\cos\sqrt{h/2}} - 1\right). \tag{14}$$

Let us consider the behavior of the director at weak magnetic fields ($h \ll 1$). The first order expansion of Eq. (14) in terms of small h gives

$$\varphi(\tilde{z}) = \frac{1}{8}\mu(\gamma-1)(1-4\tilde{z}^2)\left(1 + \frac{h}{24}(5-4\tilde{z}^2)\right). \tag{15}$$

Expression (15) describes the small deviations of the director from the unperturbed state at weak shear rates and magnetic fields (see Figs. 5 and 6). The solution in the middle of the layer (at $\tilde{z}=0$) is given by

$$\varphi_0 = \tfrac{1}{8}\mu(\gamma-1)\left(1 + \frac{5}{24}h\right). \tag{16}$$

Expression (16) at $\gamma \neq 1$ in weak fields describes only perturbed state of the director field. The increase of a magnetic field strength causes the growth of absolute value of the director angle. In flow aligning NLC ($\gamma \geq 1$) the angle φ_0 is positive and in nonflow aligning NLC $(0 \leq \gamma < 1)\varphi_0$ is negative due to positiveness of μ and h.

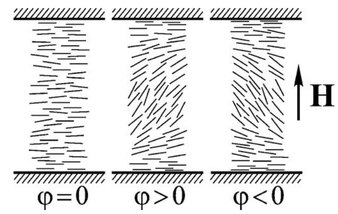

FIG. 2. The basic types of director field deformations in static Fréedericksz transition without shear flow [configuration (A)].

As it is seen from Eq. (16) in the magnetic field absence ($h=0$) the director angle φ_0 can be expressed as

$$\varphi_0 = \tfrac{1}{8}\mu(\gamma-1). \tag{17}$$

Let us notice, that the orientation of the director inside a layer is determined by the velocity gradient μ and the ratio of nematic viscous coefficients γ. In the case of shear flow of an unbounded nematic the angle of the director rotation depends on viscous coefficients only [see Eq. (9)].

3. Strong magnetic fields

In strong magnetic fields ($h \gg h_c$, $\mu \ll h$) for $\varphi_0 > 0$ from Eq. (11) we obtain

$$\sqrt{\frac{h}{2}} = \int_0^{\varphi_0} \sqrt{1 + k\tan^2\varphi}\, d\varphi.$$

Assuming $\varphi_0 = \pi/2 - \epsilon$, $\epsilon \ll 1$, we can write

$$\varphi_0 = \frac{\pi}{2} - 2\sqrt{k}e^{-\sqrt{h/2k}+\sqrt{(1-k)/k}\,\arctan\sqrt{(1-k)/k}} \text{ for } k \leq 1,$$

$$\varphi_0 = \frac{\pi}{2} - 2\sqrt{k}e^{-\sqrt{h/2k}-\sqrt{(k-1)/k}\,\operatorname{arctanh}\sqrt{(k-1)/k}} \text{ for } k \geq 1. \tag{18}$$

Equations (18) determine the asymptotic behavior of positive φ_0 values at large magnetic field strength for various anisotropy of elasticity (see Figs. 3–6).

4. Numerical simulation

The results of Eq. (11) numerical simulation are shown in Figs. 3–6. In static Fréedericksz transition without shear flow ($\mu=0$), besides the trivial solution (Fig. 2, $\varphi=0$), existing for arbitrary values of a magnetic field strength, there are two nontrivial symmetric solutions at $h \geq h_c$ [1,2]. They describe the perturbed state of the director (Fig. 3, solid line). One of the solutions corresponds to the counter-clockwise rotation of the director (Fig. 2, $\varphi > 0$), the other—to the clockwise one (Fig. 2, $\varphi < 0$). Hereinafter the critical value of a square of dimensionless magnetic field strength in static Fréedericksz transition [see Eq. (12)] is designated by means of h_c.

As we have noticed previously, the presence of shear flow ($\mu \neq 0$) does not change the threshold character of a transition for $\gamma=1$ only, that corresponds to the equality of the magnitudes of rotary viscous coefficients. The value of the critical field h_c in this case is the same [see Eq. (12)], as well

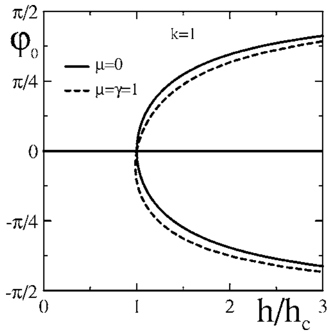

FIG. 3. Dependence of the angle φ_0 of the director rotation on the reduced square of magnetic field strength h/h_c in Fréedericksz transition [configuration (A)] at isotropy of elasticity ($k=1$). Solid line corresponds to the solutions without shear flow ($\mu=0$); dashed line describes the director behavior of a flow aligning NLC ($\gamma=1$) under shear flow influence ($\mu=1$).

as in the static transition [1,2], but the symmetry of nontrivial solutions is broken now (Fig. 3, dashed line). The presence of shear flow results in the absence of Eq. (7) invariance under the replacement $\varphi \to -\varphi$. The trivial solution still exists for arbitrary values of h. The branches of solutions corresponding to the nonuniform state of the director are displaced downward in accordance with expression (13) describing the asymmetry of the solutions close to the bifurcation point (see Figs. 3 and 4). The values of the angle φ_0 in the top branch are less on the magnitude than the values belonging to the bottom branch and corresponding to the same h. Besides from Eq. (13) it follows, that the derivative $\partial\varphi/\partial h$ at the critical point h_c is positive and finite, therefore the transition to the positive branch of solutions ($\varphi_0>0$) is carried out as the second order phase transition, and the transition to the negative branch ($\varphi_0<0$) is the first order one.

The case $\mu=k=1$ and $\gamma=-1$ was considered in Ref. [8]. The value $\gamma<0$ of the reactive parameter chosen by authors [8], is not confirmed in the experiments [1–4,9]. It is possible to assume that only NLC with disklike molecules has the values $\gamma<0$ [10–14]. Equation (7) in the case $\gamma=-1$ has no solutions describing the undistorted orientation of the director in the system (see Fig. 6 in Ref. [8]). One of the branches

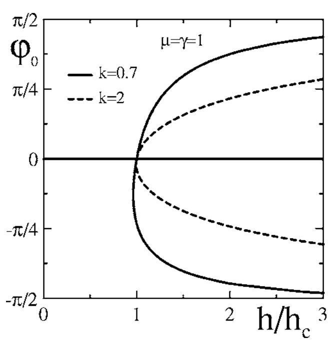

FIG. 4. Dependence of the angle φ_0 of the director rotation on the reduced square of magnetic field strength h/h_c in Fréedericksz transition [configuration (A)] for flow aligning NLC ($\gamma=1$) with two values of elastic anisotropy ($k=0.7,2$) under shear flow influence ($\mu=1$).

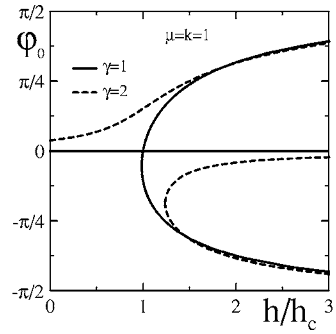

FIG. 5. Dependence of the angle φ_0 of the director rotation on the reduced square of magnetic field strength h/h_c in Fréedericksz transition [configuration (A)] for flow aligning NLC with elastic isotropy ($k=1$) and two values of reactive parameter ($\gamma=1,2$) under shear flow influence ($\mu=1$).

of the solution in Ref. [8] is continuous and exists for arbitrary values of h. It locates in the lower half-plane corresponding to the negative angles of the director rotation. Two other branches of the solution correspond to the positive values of the director angle and are situated in the upper half-plane. The bifurcation point in Ref. [8] is born at larger values of a magnetic field strength, than in the case $\gamma=1$.

Let us now consider the modification of Eq. (11) solutions with the variation of Frank elastic constants and rotary viscous coefficients. Elastic anisotropy caused by the greater magnitude of bend elastic constant than splay constant ($k<1$) leads to increase of the director rotation angle (Fig. 4, solid line). In the opposite case ($k>1$) the angle of the director rotation (Fig. 4, dashed line) decreases.

Our calculations demonstrate the "smoothing" of the phase transition (see Figs. 5 and 6) at the reactive parameter $\gamma\neq 1$, i.e., the shear flow of arbitrary intensity aligns the director at some nonzero angle to the direction of flow. The trivial solution describing a nonperturbed configuration of a director field disappears. Here the shear flow plays the same role as the external field in Landau theory of second order phase transitions.

In flow aligning NLC ($\gamma\geqslant 1$) the continuous branch of the solutions in the upper half-plane ($\varphi_0>0$) appears; it exists at arbitrary h (Fig. 5, dashed line). It is possible to explain by

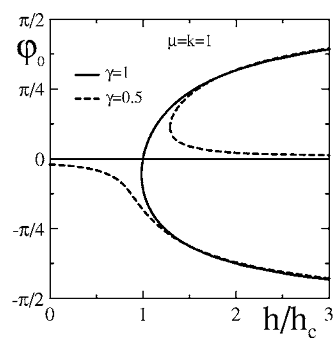

FIG. 6. Dependence of the angle φ_0 of the director rotation on the reduced square of magnetic field strength h/h_c in Fréedericksz transition [configuration (A)] for flow aligning ($\gamma=1$) and nonflow aligning NLC ($\gamma=0.5$) with elastic isotropy ($k=1$) under shear flow influence ($\mu=1$).

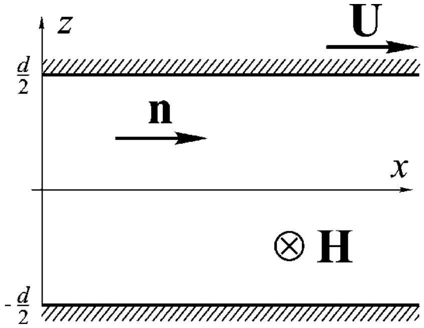

FIG. 7. Orientation of a nematic layer between moving plates in magnetic field **H** [configuration (B)].

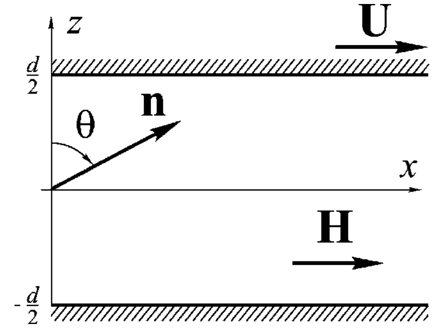

FIG. 8. Orientation of a nematic layer between moving plates in magnetic field **H** [configuration (C)].

the fact that in the absence of a magnetic field the shear flow aligns the director under a positive angle to the direction of flow [see Eq. (17)]. Imposing of a field and the subsequent increase of its strength [see Eq. (16)] leads to the growth of positive φ_0 values. Moreover, two branches of the solutions arise in the lower half-plane ($\varphi_0<0$). The bifurcation occurs at the values h larger than h_c.

In nonflow aligning NLC ($0\leqslant\gamma<1$) the continuous branch of solutions belongs to the lower half-plane ($\varphi_0<0$) (Fig. 6, dashed lines). According to Eq. (16), in this case φ_0 is negative, and increase of a magnetic field strength provides its further reduction. The appearance of the nontrivial solutions describing the counter-clockwise rotation of the director ($\varphi_0>0$), as well as in the previous case, takes place at values h larger than in the static Fréedericksz transition.

B. Configuration (B)

Let us direct the magnetic field $\mathbf{H}=(0,H,0)$ perpendicularly to the velocity of flow $\mathbf{v}=(v_x(z),0,0)$ in a plane of a nematic layer how it is shown in Fig. 7.

We assume conditions of rigid planar coupling for the director on the boundaries of the layer. In the absence of a magnetic field and shear flow the director inside the layer is uniform and directed along the x axis. As it is shown in the configuration (A), without magnetic field a shear flow aligns the director under some angle to a flow in the xOz plane [see Eq. (17)]. Imposing a magnetic field in a plane of a layer perpendicular to the direction of flow for reactive parameter $\gamma=-\gamma_2/\gamma_1\neq 1$ leads to three-dimensional deformations of a director field.

Let us consider the problem for the planar director field. We choose $\gamma=1$, as only in this case the director is aligned by flow along the x axis. In the presence of shear flow and magnetic field it is possible to search the director $\mathbf{n}$ in the following form:

$$\mathbf{n}=(\cos\psi(z),\sin\psi(z),0), \tag{19}$$

where $\psi(z)$ is the angle between the director $\mathbf{n}$ situated in the plane of the plates, and the x axis.

By substituting Eq. (19) into the equation of the director motion (4) for the stationary case ($\partial\mathbf{n}/\partial t=0$) we obtain

$$K_2\frac{d^2\psi}{dz^2}=-\frac{\chi_a H^2}{2}\sin 2\psi, \tag{20}$$

$$(\gamma_1+\gamma_2)\frac{du}{dz}\cos\psi=0. \tag{21}$$

Equation (21) satisfies identically for $\gamma=1$ (i.e., $\gamma_1=-\gamma_2$) and Eq. (20) for dimensionless coordinate $\tilde{z}=z/d$ can be written as follows:

$$\psi''=-h\sin 2\psi \tag{22}$$

with boundary conditions

$$\psi(-1/2)=\psi(1/2)=0, \tag{23}$$

where $h=\chi_a H^2 d^2/(2K_2)$.

The solution of Eq. (22) in the middle of the layer has the form

$$\sqrt{h/2}=K(\sin\psi_0), \tag{24}$$

where $K(\sin\psi_0)$ is the complete elliptic integral of the first kind [17], and $\psi_0=\psi(0)$ is the angle of the director orientation in the middle of the layer.

Equation (24) describing the Fréedericksz transition at presence of shear flow ($\mu\neq 0$) coincides with the well-known equation for the static case ($\mu=0$) [3,4]. Except for the solution $\psi_0=0$ corresponding to the nonperturbed state of the director field, Eq. (22) supposes two symmetric solutions at $h\geqslant h_c$ with positive or negative values of an angle of director rotation. In the considered approach the angle ψ_0 of director $\mathbf{n}$ rotation does not depend on the velocity of relative movement of plates.

In Ref. [8] the authors assumed $\gamma_1=\gamma_2$, that corresponds to reactive parameter $\gamma=-1$. In this case Eq. (21) has only the trivial solution $\psi=\pi/2$ which is not satisfied, however, we have the boundary conditions (23). In this case it is impossible to use the director profile $\mathbf{n}$ in form (19) due to three-dimensional deformations of the director field.

C. Configuration (C)

Let the magnetic field $\mathbf{H}=(H,0,0)$ be directed parallel to the velocity of shear flow $\mathbf{v}=(v_x(z),0,0)$ as is shown in Fig. 8. On the boundaries of a layer we use the conditions of rigid homeotropic coupling. For that case in the absence of a magnetic field and flow the director in the layer is uniform and directed along the z axis. In the presence of flow and a magnetic field the director $\mathbf{n}$ can be written in the form

$$\mathbf{n} = (\sin\,\theta(z), 0, \cos\,\theta(z)), \tag{25}$$

where $\theta(z)$ is the angle between the director $\mathbf{n}$ and the z axis.

We use the approximation of linear distribution for the velocity field in the layer [8,16] $\mathbf{v}=(Uz/d,0,0)$, here U is the velocity of the top plate, d is the thickness of a layer.

Equation of the director motion (4) in a stationary case $(\partial\mathbf{n}/\partial t=0)$ with the help of Eq. (25) for dimensionless coordinate $\tilde{z}=z/d$ becomes

$$f_C(\theta)\frac{d^2\theta}{d\tilde{z}^2} + \frac{1}{2}\frac{df_C(\theta)}{d\theta}\left(\frac{d\theta}{d\tilde{z}}\right)^2 = -h\sin 2\theta - \mu(1+\gamma\cos 2\theta) \tag{26}$$

with boundary conditions

$$\theta(-1/2) = \theta(1/2) = 0, \tag{27}$$

where $f_C(\theta)=\sin^2\theta + k\cos^2\theta$. The dimensionless parameters h, μ, k, and γ are defined in the same way as in configuration (A).

As it is seen from Eq. (26) the uniform state of the director corresponding to trivial solution $\theta=0$ is possible at the presence of flow and magnetic field for the reactive parameter $\gamma=-1$ only. For NLC with rodlike molecules the uniform solution is impossible, because the reactive parameter is positive.

Let us find nonuniform solutions for a director field. For this purpose we multiply Eq. (26) by $d\theta/d\tilde{z}$ and integrate it twice. The stationary solutions for the perturbed state of the director in the middle of the NLC layer is possible to obtain from the following integral equation:

$$\frac{1}{2} = \pm\int_0^{\theta_0}\sqrt{\frac{f_C(\theta)}{\Phi_C(\theta,\theta_0)}}d\theta, \tag{28}$$

where $\Phi_C(\theta,\theta_0)=h(\cos 2\theta - \cos 2\theta_0)+2\mu(\theta_0-\theta)+\mu\gamma \times(\sin 2\theta_0 - \sin 2\theta)$. Here $\theta_0=\theta(0)$ is the angle of the director orientation in the middle of the layer. The plus sign in Eq. (28) corresponds to clockwise rotation of the director $(\theta_0>0)$, and the minus sign corresponds to counterclockwise rotation of the director $(\theta_0<0)$.

1. Weak flows and magnetic fields

In this configuration the trivial solution $\theta=0$ of Eq. (26) for arbitrary values of h exists only in the absence of flow, i.e., at $\mu=0$. We consider the case of weak flow $(\mu/k \ll 1)$ when the angle $\theta \ll 1$. The first order expansion of Eq. (26) in terms of small θ gives

$$k\theta'' + 2h\theta = -\mu(1+\gamma),$$

and using boundary conditions (27), we find

$$\theta(\tilde{z}) = \frac{\mu(1+\gamma)}{2h}\left(\frac{\cos(\sqrt{2h/k}\tilde{z})}{\cos\sqrt{h/2k}} - 1\right). \tag{29}$$

In weak magnetic fields $(h \ll 1)$ the linear order approximation of Eq. (29) has the form

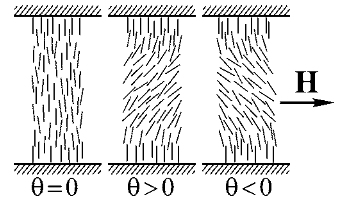

FIG. 9. The basic types of director field deformations in static Fréedericksz transition without shear flow [configuration (C)].

$$\theta(\tilde{z}) = \frac{1}{8k}\mu(1+\gamma)(1-4\tilde{z}^2)\left(1+\frac{h}{24k}(5-4\tilde{z}^2)\right). \tag{30}$$

Expression (30) describes small distortions of the director angle at weak flows and magnetic fields (Figs. 10–12). From Eq. (30) for the angle of director rotation in the middle of a layer we obtain

$$\theta_0 = \frac{\mu(1+\gamma)}{8k}\left(1+\frac{5h}{24k}\right). \tag{31}$$

As it is seen from Eq. (31) the director angle is always positive in weak fields for reactive parameter $\gamma \geqslant 0$. It means, that it is possible to expect the identical character of "smoothing" of the transition in flow aligning $(\gamma \geqslant 1)$ NLC and nonflow aligning $(0 \leqslant \gamma < 1)$ NLC (see Fig. 12). Furthermore, the growth of θ_0 can be caused either by the increase of magnetic field strength h or gradient of shear rate μ (see Fig. 10).

2. Strong magnetic fields

In strong fields $(h \gg h_c, \mu \ll h)$ for $\theta_0>0$ from Eq. (28) we obtain

$$\sqrt{\frac{h}{2}} = \int_0^{\theta_0}\sqrt{k+\tan^2\theta}\,d\theta. \tag{32}$$

Assuming $\theta_0=\pi/2-\epsilon$, $\epsilon \ll 1$, we find

$$\theta_0 = \frac{\pi}{2} - \frac{2}{\sqrt{k}}e^{-\sqrt{h/2}-\sqrt{1-k}\,\operatorname{arctanh}\sqrt{1-k}} \quad \text{for } k \leqslant 1,$$

$$\theta_0 = \frac{\pi}{2} - \frac{2}{\sqrt{k}}e^{-\sqrt{h/2}+\sqrt{k-1}\,\arctan\sqrt{k-1}} \quad \text{for } k \geqslant 1. \tag{33}$$

Equations (33) describe the asymptotic behavior of positive θ_0 values at large magnetic field strength for various anisotropy of elasticity (see Figs. 10–12).

3. Numerical simulation

Results of numerical solution of Eq. (28) are shown in Figs. 10–12.

In considered configuration in the absence of a flow $(\mu=0)$, as well as in the configuration (A), in addition to the trivial solution (Fig. 9, $\theta=0$), existing at arbitrary values of magnetic field strength, there are two symmetric solutions [1,2]. They describe the perturbed state of the director and appear at critical field strength $h_c=\pi^2/2$ (Fig. 10, solid

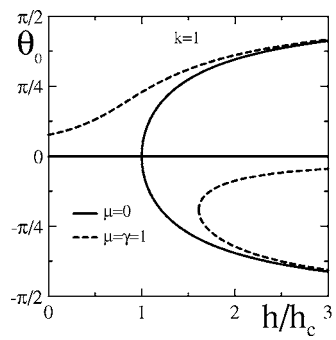

FIG. 10. Dependence of the angle θ_0 of the director rotation on the reduced square of magnetic field strength h/h_c in Fréedericksz transition [configuration (C)] at isotropy of elasticity ($k=1$). Solid lines correspond to the solutions without shear flow ($\mu=0$); dashed lines describe the transition in flow aligning NLC ($\gamma=1$) under shear flow influence ($\mu=1$).

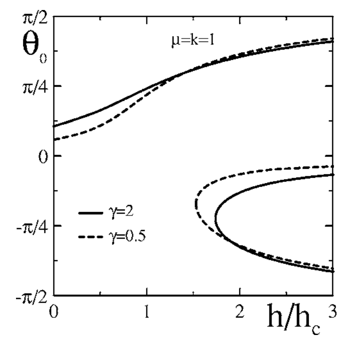

FIG. 12. Dependence of the angle θ_0 of the director rotation on the reduced square of magnetic field strength h/h_c in Fréedericksz transition [configuration (C)] for flow aligning ($\gamma=2$) and nonflow aligning NLC ($\gamma=0.5$) at presence of shear flow ($\mu=1$) and isotropy of elasticity ($k=1$).

lines). One of the solutions corresponds to clockwise rotation of the director (Fig. 9, $\theta>0$), the other to counter-clockwise rotation (Fig. 9, $\theta<0$).

The presence of shear flow ($\mu\neq 0$) in configuration (C) leads to the impossibility of existence of undistorted director conformation for arbitrary h values because Eq. (26) has no $\theta=0$ solution in this case. The behavior of NLC with reactive parameter $\gamma=1$ and equal values of Frank elastic constants ($k=1$) is shown in Fig. 10 (dashed lines). In the considered configuration the phase transition is smoothed stronger in more intensive flow (i.e., at higher μ), in accordance with Eq. (31).

The violation of elastic isotropy corresponding to the smaller bend module than splay module ($k<1$), leads to increase of positive values of the angle of the director rotation (Fig. 11, solid lines). Such a behavior of the director is consistent with the preceding results [see Eq. (31)]. The bifurcation point is shifted in area of smaller h values, than in the isotropic case (see Fig. 10). At $k>1$ the opposite tendency (Fig. 11, dashed lines) takes place: the decrease of the director rotation angle for the solutions lying in the upper half-plane ($\theta_0>0$). The solutions corresponding to negative θ_0 at $k>1$ are found out in stronger fields, than at $k\leqslant 1$. For reactive parameter $\gamma\geqslant 1$ (a case of flow aligning NLC) the calculations show the increase of positive values of θ_0 in weak fields [see Eq. (31)]. Further increase of a field strength leads to appearance of two branches of solutions in the lower half-plane ($\theta_0<0$) (Fig. 12, solid lines).

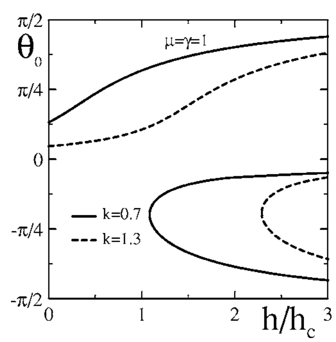

FIG. 11. Dependence of the angle θ_0 of the director rotation on the reduced square of magnetic field strength h/h_c in Fréedericksz transition [configuration (C)] for flow aligning NLC ($\gamma=1$) with two values of anisotropy of elasticity ($k=0.7, 1.3$) at presence of shear flow ($\mu=1$).

In nonflow aligning NLC ($0\leqslant\gamma<1$) at small h the reduction of positive θ_0 values [see Eq. (31)] is observed. Solutions lying in the lower half-plane at $\gamma<1$ (Fig. 12, dashed lines) appear in weaker fields than at $\gamma>1$. The character of "smoothing" of phase transition in configuration (C) for flow aligning and nonflow aligning NLC is identical. Due to $\gamma>0$ for rodlike nematics the sign of the angle of the director orientation in shear flow does not depend on the reactive parameter [see Eq. (31)], in contrast to the configuration (A) [see Eq. (17)].

In Ref. [8] authors supposed that $\gamma=-1$, therefore Eq. (26) had the trivial solution, and so they obtained the essentially different behavior of the director angle θ_0 in external field for this configuration (see Fig. 5 in Ref. [8]).

IV. CONCLUSIONS

In the present paper we have examined the influence of shear flow on the Fréedericksz transition induced by magnetic field in a planar layer of a nematic liquid crystal. Conditions of rigid director coupling on the boundaries and linear distribution of a velocity field inside the nematic layer have been assumed. We have obtained stationary solutions for a planar director field **n** in the middle of the layer for different values of Frank elastic modules and viscous coefficients. We have considered both flow aligning ($\gamma\geqslant 1$) and nonflow aligning ($0\leqslant\gamma<1$) NLC with rodlike molecules for three basic orientations of NLC layer in magnetic field. Configuration (A) corresponds to magnetic field which is perpendicular to the plane of the layer and the direction of flow. In configuration (B) the magnetic field lying in the plane of the layer is directed perpendicularly to the flow. And, at last, the magnetic field is in the plane of the layer and parallel to the flow direction in configuration (C).

The solutions received above are applicable far from oriented plates, on distances, large compared with the dimensionless length ζ determined by the relation [15]

$$\zeta^2 = \frac{1}{2\sqrt{h^2 + \mu^2(\gamma^2 - 1)}},$$

which coincides with magnetic coherence length in the limits of strong magnetic fields or for equal absolute values of NLC rotary viscous coefficients [1,2]. The presence of shear flow in configuration (A) leads to symmetry breaking of perturbed state solutions of the director. The existence of nonperturbed field of the director at arbitrary values of magnetic field strength is possible at reactive parameter $\gamma=1$ only. In configuration (B) there are no changes in the Fréedericksz transition associated with imposing of shear flow, at $\gamma=1$ only. In the considered approach the solution does not depend on relative velocity of the plates. The situation varies in configuration (C), where the presence of shear does not suppose the existence of nonperturbed director field for arbitrary values of magnetic field strength in nematics with rodlike molecules. At any allowable values of reactive parameter the "smoothing" of phase transition takes place.

ACKNOWLEDGMENTS

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